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EXACT MULTIMONOPOLE SOLUTIONS IN THE BOGOMOLNY-PRASAD-SOMMERFIELD LIMIT

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ABSTRACT

A systematic method for generating axially symmetric multimonopole solutions is presented. The Bogomolny-Prasad-Sommerfield one monopole and a new doubly charged monopole are obtained via Harrison's Bäcklund transformation.

АННОТАЦИЯ

Развивается метод получения аксиально симметричных многомонопольных решений. Единичный монополь Богомольного-Прасада-Соммерфильда и новый двукратно заряженный монополь генерируется путем преобразования Бэклунда по Гаррисону.

KIVONAT

Axiálisan szimmetrikus multimonopolus megoldások generálására alkalmas módszert fejlesztünk ki. A Bogomolny-Prasad-Sommerfield egy monopólust és egy uj kétszeres töltésü monopólust Harrison Bäcklund transzformációja segitségével generáljuk.

There has been a recent upsurge of interest in the theory of magnetic monopoles. The existence of static multimonopole solutions in an SU(2) Yang-Mills-Higgs theory in the limit of vanishing Higgs potential [1] has been conjectured since Manton has shown that there are no long range forces between equally charged monopoles [2]. Quite recently Taubes has proved the existence of such multimonopole solutions [3]. We have shown [4] that the Bogomolny Equations for the simplest axially symmetric ansatz constructed by Manton [5] reduce to a rather well studied equation, the Ernst equation of general relativity [6]. For the Ernst equation there exist systematic solution generating methods, such as the inverse scattering method and the Bäcklund transformations /BT/. In Ref. [4] it was shown how to generate the Bogomolny-Prasad-Sommerfield /BPS/ one monopole /1MP/ solution using Bäcklund transformations. Given the fact the BT's can be iterated algebraically, it seems reasonable to hope, that in this way one can actually generate multimonopole solutions. The aim of this paper is to show that this is indeed the case, and we display below, using our methods, an explicit two monopole /2MP/ solution. Our solution is axially symmetric and corresponds to two superimposed monopoles located at the origin.

Manton's Ansatz in polar coordinates is

$$\begin{aligned} A_t^a &= (0, \phi_1, \phi_2) & A_\varphi^a &= -(0, \eta_1, \eta_2) \\ A_z^a &= -(W_1, 0, 0) & A_\varrho^a &= -(W_2, 0, 0) \end{aligned} \tag{1}$$

where $x_1 = \varrho\cos\varphi$, $x_2 = \varrho\sin\varphi$, and η_i, ϕ_i, W_i are functions of ϱ, z only. In Ref. [4] it was shown that the Bogomolny equations reduce to

$$Re\,\epsilon\,\Delta\epsilon - (\nabla\epsilon)^2 = 0 \tag{2}$$

where $\epsilon = f + i\psi$, $\Delta = \partial_\rho^2 + \rho^{-1}\partial_\rho + \partial_z^2$. The equation (2) is the so-called Ernst equation. The functions ϕ_i, η_i, W_i may be obtained from ϵ in the following way

$$\phi_1 = f^{-1}\psi_{,z} \qquad \eta_1 = -\rho f^{-1}\psi_{,\rho} \qquad W_1 = -f^{-1}\psi_{,z} \tag{3}$$

$$\phi_2 = -f^{-1}f_{,z} \qquad \eta_2 = \rho f^{-1}f_{,\rho} \qquad W_2 = -f^{-1}\psi_{,\rho}$$

For details we refer to [4] . Now for eq. (2) there are various solution generating (group-theoretic [7] or soliton-theoretic) techniques: Bäcklund transformations found by Harrison /HB/ [8] and Neugebauer /NB/ [9] , the inverse scattering method of Belinsky and Zakharov /BZ/ [10] and the integral equation approach devised by Hauser and Ernst [11] . Here we apply the HB transformations to generate the 1MP and the 2MP solutions from a suitably chosen ground state /vacuum/.

To apply the HB transformation one defines from a known solution $\epsilon = f + i\psi$ of eq. (2) the quantities

$$M_1^0 = \frac{1}{2} f^{-1}\epsilon_{,1} \;;\quad M_2^0 = \frac{1}{2} f^{-1}\epsilon^*_{,1} \;;\quad N_1^0 = \frac{1}{2} f^{-1}\epsilon^*_{,2} \;;$$

$$N_2^0 = \frac{1}{2} f^{-1}\epsilon_{,2} \;;\quad \epsilon_{,1} = \frac{\partial\epsilon}{\partial\zeta_1} \;;\; \epsilon_{,2} = \frac{\partial\epsilon}{\partial\zeta_2} \;;\; \zeta_1 = \rho + iz \;;\; \zeta_2 = \rho - iz \tag{4}$$

and solves the total Riccati equation for the pseudopotential $q(\zeta_1, \zeta_2)$:

$$dq = \left[(M_2^0 - M_1^0)q + \gamma(w)(M_2^0 - M_1^0 q^2)\right]d\zeta_1 + \\ + \left[(N_1^0 - N_2^0)q + \gamma^{-1}(w)(N_1^0 - N_2^0 q^2)\right]d\zeta_2 \tag{5}$$

where $\gamma(w) = \left[(w - i\zeta_2)(w + i\zeta_1)^{-1}\right]^{1/2}$, w being an arbitrary constant.

The new /transformed/ M_i's are given in terms of M_i^0, q and $\gamma(w)$ as

$$M_1^{(1)} = H(q,\gamma)M_1^0 = -q\frac{1+\gamma q}{\gamma+q}M_1^0 - q\frac{\gamma^2-1}{\gamma+q}\frac{1}{4g} \tag{6}$$

$$M_2^{(1)} = H(q,\gamma)M_2^0 = -\frac{\gamma+q}{q(1+\gamma q)}M_2^0 - \frac{1}{4g}\frac{\gamma^2-1}{1+\gamma q}$$

However, before entering into the details of generating the monopole solutions, we define the action of the so-called Neugebauer-Kramer mapping /I/ [12] acting on the M_i's which will be frequently used in this paper:

$$IM_1 = -M_2 + \frac{1}{4g}\ , \qquad IM_2 = -M_1 + \frac{1}{4g}\ ,$$
$$IN_1 = -N_1 + \frac{1}{4g}\ , \qquad IN_2 = -N_2 + \frac{1}{4g} \tag{7}$$

It is an advantage of these transformations that they act on M_i's as these are in direct connection – via (3) – with the fields ϕ_i, η_i and W_i of our interest. The gauge invariant length of the Higgs field, $\phi^2 = \phi_1^2 + \phi_2^2$ is

$$\phi^2 = \frac{f_{,z}^2 + \psi_{,z}^2}{f^2} = 4(M_1-N_2)(N_1-M_2) \tag{8}$$

The magnetic charge /n/ can be immediately calculated from ϕ^2

$$n = \lim_{r\to\infty}\frac{1}{8\pi}\int_{r=\text{const.}} dS_i\, \partial_i|\phi|^2 \tag{9}$$

since the Bogomolny equations are satisfied. The most important property of the HB transformations is, that once eq. (5) for the pseudopotential q is solved, the BT's can be iterated algebraically. In fact, let both (q_1, γ_1) ; (q_2, γ_2) satisfy eq. (5) with the same M_i^0, N_i^0 but different constants w_1, w_2. As it was shown by Cosgrove [13], the pseudopotential for the second HB transformation q' is given by

$$q' = \frac{\gamma_2(1-\gamma_1^2)q_1 - \gamma_1(1-\gamma_2^2)q_2 - (\gamma_2^2-\gamma_1^2)q_1 q_2}{q_1\left[(\gamma_1^2-\gamma_2^2) + \gamma_1(1-\gamma_2^2)q_1 - \gamma_2(1-\gamma_1^2)q_2\right]} \tag{10}$$

This q' satisfies (5) with $\gamma = \gamma_2$ and M_i^0, N_i^0 replaced by $M_i = H(q_1\gamma_1)M_i^0$, $N_i = H(q_1\gamma_1)N_i^0$.

In Ref. [4] the 1MP solution was generated from a natural ground state, where $\phi^2 = 1$ /Higgs vacuum/, with the aid of the product transformation HI. /We remark here that the transformations IH and HI lead to the same results provided the parameters are chosen appropriately [13]./ Here we follow the same line of attack, namely, we apply the iterated transformations IHIH to the solution

$$f = e^z, \quad \psi = 0 \tag{11}$$

where $M_1^0 = M_2^0 = -\frac{i}{4}$, $N_1^0 = N_2^0 = \frac{i}{4}$

As it is evident from (3), (11) is a Higgs vacuum too. We note that in Ref. [4] a different ground state was used to generate the 1MP, the connection between these two states will be discussed elsewhere [14].

It is easy to integrate eq. (5) for the seed solution (11), the result is

$$q = -\operatorname{th}\left(\frac{R_w}{2} - \beta\right) \tag{12}$$

where $R_w = \sqrt{(w-z)^2 + \rho^2}$ and β is the constant of integration. The transformation IH acts on M_i^0 's, N_i^0's as

$$IH\,M_1^0 = \frac{\gamma+q}{1+\gamma q}\left(q^{-1}M_2^0 + \frac{\gamma}{4\rho}\right)$$
$$IH\,N_2^0 = \frac{\gamma+q}{1+\gamma q}\left(q\,N_2^0 + \frac{1}{\gamma 4\rho}\right) \tag{13}$$

The following conditions

$$M_1^* = N_1 \quad ; \quad M_2^* = N_2 \tag{14}$$

ensure that the new solution is real. To achieve this we choose the parameters β, w in such a way that the pseudopotential be real / $q^* = q$ /, i.e. w and β should be real numbers. From this it follows immediately that

$$\gamma^* = \gamma^{-1} \quad \text{, and} \quad \left|\frac{\gamma+q}{1+\gamma q}\right| = 1. \tag{15}$$

Choosing $\beta = 0$, from (13) and (8) we obtain the Φ^2 of the well-known BPS 1MP

$$\Phi^2 = \left(\coth R_w - \frac{1}{R_w}\right)^2 \tag{16}$$

We now proceed to iterate the IH transformation twice to generate the 2MP. Since, in a sense, I and H commute / as it was mentioned earlier /, and $I^2=1$ which is easily verified from (7) , in fact, IHIH= H'H . Therefore, to carry out the next step of the iteration amounts to replacing q in (6) by q' from (10) and M_i^0 by $H(q_1\,\gamma_1)\,M_i^0$. After some algebra we get

$$M_1^{(2)} = H'HM_1^0 = q'q_1 \left(\frac{\gamma_1 q_1 - \gamma_2 q_2}{\gamma_2 q_1 - \gamma_1 q_2} M_1^0 + \frac{1}{4g} \frac{\gamma_1^2 - \gamma_2^2}{q_1\gamma_2 - \gamma_1 q_2} \right)$$

$$N_2^{(2)} = q'q_1 \left(\frac{\gamma_2 q_1 - \gamma_1 q_2}{\gamma_1 q_1 - \gamma_2 q_2} N_2^0 + \frac{\gamma_2^2 - \gamma_1^2}{4g\, \gamma_1 \gamma_2 (\gamma_1 q_1 - \gamma_2 q_2)} \right) \tag{17}$$

The reality conditions (14) can be satisfied in two ways:

$$\gamma_i^* = \gamma_i^{-1} \;;\quad q_i^* = q_i^{-1} \qquad (i=1,2) \tag{18a}$$

$$\gamma_1^* = \gamma_2^{-1} \;;\quad q_1^* = q_2^{-1} \tag{18b}$$

In both cases $|q'q_1| = 1$. (18a) implies β_i, w_i to be real, which could describe two monopôles located at different points of the z-axis. Nevertheless, for these solutions ϕ^2 is singular corresponding to infinite energy. This supports the results of Refs [15,16] stating that there are no axially symmetric multimonopole solutions, unless they are located at a single point.

The conditions (18b) are satisfied by the choice: $w_1 = w_2^* = i\alpha$; $\beta_1 = iB$; $\beta_2 = -i\,(B + \pi/2)$. To guarantee the appropriate behaviour on the z-axis we have to choose $B = \frac{\alpha}{2} - \frac{\pi}{4}$. To calculate explicitely ϕ^2 it proves advantageous to use oblate spheroidal coordinates

$$\sqrt{(1-\xi^2)(1+\eta^2)} + i\,\xi\eta = \frac{g + iz}{\alpha}\;;\; -1 \le \xi \le 1;\; 0 \le \eta < \infty \tag{19}$$

in terms of which after some straightforward but painful algebra we got for our 2MP solution

$$\Phi^2 = J^{-1}\Big\{\Big[(1+\eta^2)\cos\alpha\xi\big(\alpha[\eta^2+\xi^2]\cos\alpha\xi - 2\xi\sin\alpha\xi\big) +$$

$$+ (1-\xi^2)\cosh\alpha\eta\big(\alpha[\eta^2+\xi^2]\cosh\alpha\eta - 2\eta\sinh\alpha\eta\big)\Big]^2 +$$

$$+ 4\Big[\xi(1+\eta^2)\cos\alpha\xi\sinh\alpha\eta - \eta(1-\xi^2)\cosh\alpha\eta\sin\alpha\xi\Big]^2\Big\}$$

where

$$J = \alpha^2(\eta^2+\xi^2)^2\Big[(1-\xi^2)\cosh^2\alpha\eta - (\eta^2+1)\cos^2\alpha\xi\Big]^2 \qquad (20)$$

This is the main result of our paper. To illustrate that (20) really describes a doubly charged monopole, we present the behaviour of our Φ^2 in different characteristic regions. First, we turn our attention to the asymptotic region $\sqrt{\varsigma^2+z^2} = r \longrightarrow \infty$:

$$\Phi^2 = \Big[1 - \frac{2}{r}\Big(1 + \frac{\alpha}{r}\sin\vartheta\Big) + O\Big(\frac{1}{r^3}\Big)\Big]^2 \qquad (21)$$

where ϑ is the usual polar angle. Since the coefficient of the $\frac{1}{r}$ term in Φ^2 is 4, from (9) we obtain for the magnetic charge n = 2. On the z-axis / $\xi = \pm 1$ / (20) simplifies to

$$\Phi^2 = \Big(\tanh\alpha\eta - \frac{2\eta}{\alpha(\eta^2+1)}\Big)^2 \qquad (22)$$

(22) shows that at the origin / $\eta = 0$ / $|\Phi|$ has a first order zero. On the z = 0 plane / $\xi = 0$ or $\eta = 0$ /

$$\Phi^2 = \Big(1 + \frac{2\cos\alpha\xi\,[\alpha\xi\cos\alpha\xi - \sin\alpha\xi]}{\alpha\xi\,[\sin^2\alpha\xi - \xi^2]}\Big)^2 \quad (\eta = 0) \qquad (23)$$

similarly for $\xi = 0$ in (23), one has to replace ξ by $i\eta$. It is clear from (23) that at the origin $|\phi|$ has a double zero. To ensure that on this plane ϕ^2 be free of singularities we had to fix $\alpha = \frac{\pi}{2}$.

With this choice it is not difficult to prove that the denominator in (20) does not vanish away from the z = 0 plane, guaranteeing that ϕ^2 is nowhere singular. Similarly, we have seen that there are no other zeros of ϕ^2 apart from the origin.

Although, in this paper we have presented only the explicit form of ϕ^2 it is not difficult to compute in this gauge the components of the vector potential A_μ using formulae (3), (4), (17). It immediately follows from the reality conditions (14) that A_μ is real in this gauge.

We remark that the numerical solution of Ref. [16] is in qualitative agreement with our result. In the energy density there is a bump located roughly at the ring $\xi = \eta = 0$. This and the fact that the oblate spheroidal coordinate system emerged naturally suggest that the two monopoles strongly deform the field of each other. So, it seems reasonable to expect, that the monopoles separated by a finite distance are in a sense "pancake-like". Of course, this can only be confirmed by finding such an exact solution, for which one has to abandon axial symmetry [15,16]. Nevertheless, we hope, that the techniques developped in Ref. [17] will be powerful enough to enable one to meet this challenge.

Still within Manton's Ansatz we have iterated the IH transformation three times to construct a 3MP. Properties of this solution are under study. The details of the 3MP and of our method outlined in this paper will be published elsewhere [14].

In conclusion, we have developped a method for generating axially symmetric multimonopole solutions of the Bogomolny equations and displayed the explicit form of the 2MP.

During the completion of our work we got a preprint by Ward [18] in which he obtained similar result for the 2MP solution, in particular, our formulae (22) ,(23) after suitable rescaling are identical to his expressions (8),(9).

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